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Approximate Method for Estimating Wake Vortex Strength

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An approximate method is presented for estimating the strength of slender-body wake vortices. The method is shown to yield good accuracy for the case of asymmetric vortices in the wake of a body at high angle of attack.

Nomenclature

A = area of wake vortex cross section
C_p = pressure coefficient
d = body diameter
F = vorticity flux
g = body length from which boundary-layer fluid is shed to form single vortex
k = dimensionless velocity parameter
M = Mach number
dl = elementary length vector in plane of A
$\bar{q}$ = general velocity vector
Re = Reynolds number
U = circumferential component of velocity at boundary-layer edge
u = circumferential component of velocity in boundary layer
V = freestream velocity
$\bar{w}$ = general vorticity vector
x = distance parallel to body axis
y = distance normal to surface
α = angle of attack
δ = boundary-layer thickness
θ = angle around circumference, measured from windward side meridian
ξ = angle between vortex cores and body axis
Γ = vortex strength
Γ_p = vortex strength parameter
χ = quantity associated with vortex core angle, = $\tan \xi / \tan \alpha$

Subscripts
B = from body
a = along vortex core
c = crossflow
s = separation
V = in vortex

Superscript
$*$ = critical value

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Introduction

A MAJOR problem in missile aerodynamics is posed by the separation and subsequent behavior of the boundary layers on various missile components. This is particularly so when the separated flow forms large, powerful wake vortices whose effect on downstream components can be severe. A well-known example is furnished by the vortices in the wake of a slender axisymmetric body at incidence. These vortices first appear, at

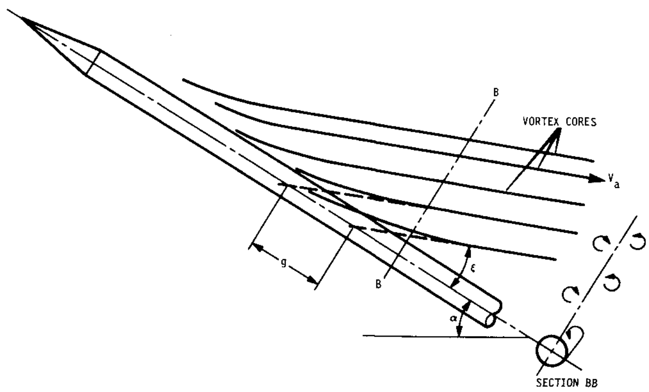

Fig. 1 Schematic of lee side vortex pattern.

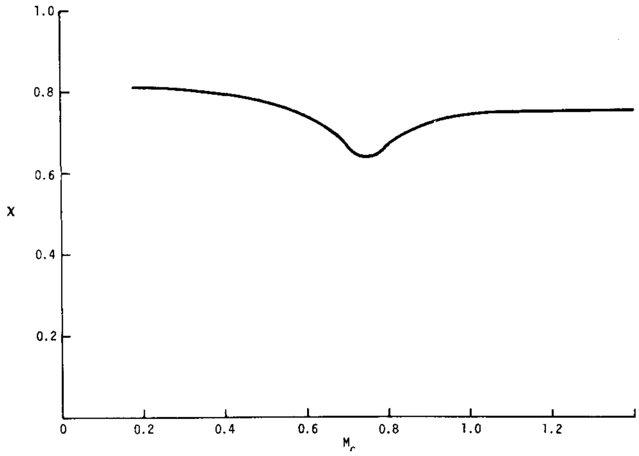

Fig. 2 Vortex core angle parameter.

about 6° angle of attack, as a symmetric pair. As angle increases the pair is ultimately joined by a third, a fourth, etc., and the pattern becomes asymmetric (Fig. 1). When this occurs, a section taken normal to the body axis through the wake resembles the two-dimensional von Kármán vortex street well-known in the literature. Each of these street vortices is formed close to the body in the same way as a symmetric vortex. When the vortex strength reaches a critical value, however, the angle between the core and the body increases and the vortex proceeds downstream, receiving no further vorticity from the boundary layer (see Fig. 1). This paper describes a general semiempirical technique for calculating the strengths of wake vortices in both the symmetric and asymmetric cases.

Now the wake vortices are supplied with vorticity from the separating body boundary layers. The net vorticity flowing into a vortex is the resultant of the fluxes in two boundary layers whose confluence is marked by the separation line. One layer is associated with the freestream; the other with the leeside vortex flow. It has been shown by Wang[1] that, at least for laminar flow, what is usually termed the separation line on a slender body need only be the locus of points where the crossflow boundary-layer profile reverses. The axial flow profile shows no reversal. Accordingly, the vorticity flux from the body will here be defined by the net flux from the crossflow and backflow boundary layers only.

It is convenient to define the net flux as a fraction, Λ, of that in the crossflow boundary layer alone. A similar fraction has been used in work on two-dimensional boundary layers,[2-4] where the value was found, empirically, to be about $\frac{1}{2}$. Considering then a vortex being formed by the flow separating over some axial distance x of the body, the net flux of vorticity, F_B, flowing into the vortex per unit time is

$$F_B = \Lambda \int_0^x \int_0^{\delta_s} u \frac{\partial u}{\partial y} dy\, dx \qquad \text{to the boundary-layer approximation}$$

or

$$F_B = \Lambda \int_0^x \frac{U_s^2}{2} dx \qquad \text{where } U_s \text{ may vary in the axial direction}$$

Turning now to the wake vortex fed by the shed boundary-layer fluid, the total flux per unit time of streamwise vorticity flowing downstream in this vortex is

$$F_V = V_a \int_A \bar{w} \cdot d\bar{A}$$

where A is the total area of the vortex cross section and V_a is the flow velocity along the vortex core. By Stokes theorem, the last integral may be replaced with the integral of tangential velocity $\bar{q}$ round a circuit enclosing the vortex, i.e., the total vortex circulation strength

$$F_V = V_a \oint \bar{q} \cdot d\bar{l} = V_a \Gamma$$

Assuming conservation of net vorticity, F_B and F_V may be equated to yield

$$V_a \Gamma = \frac{\Lambda}{2} \int_0^x U_s^2 dx \tag{1}$$

Equation (1) is an expression for the vortex strength. Γ may be found if all the other quantities are available. The only source of these quantities at present is empirical evidence. Such evidence, drawn from several sources, will be input to the equation. The resulting vortex strength will be compared with experimentally-measured values of the parameter.

Empirical Inputs

The flow pattern chosen for use of Eq. (1) is the asymmetric vortex system which appears on slender missile bodies at high angles of attack. Extensive experimental evidence is available for such flows.[5,6] Reference 5 gives data for V_a and the length g, over which the boundary layer is shed for a single vortex. Reference 6 provides data on θ_s, the circumferential separation angle. Additional experimental evidence relating θ_s and U_s is provided by Ref. 8. Finally, the calculated vortex strength may be compared with that measured through wake traverses and integrations in Ref. 5. The results of Ref. 5 may be used to show

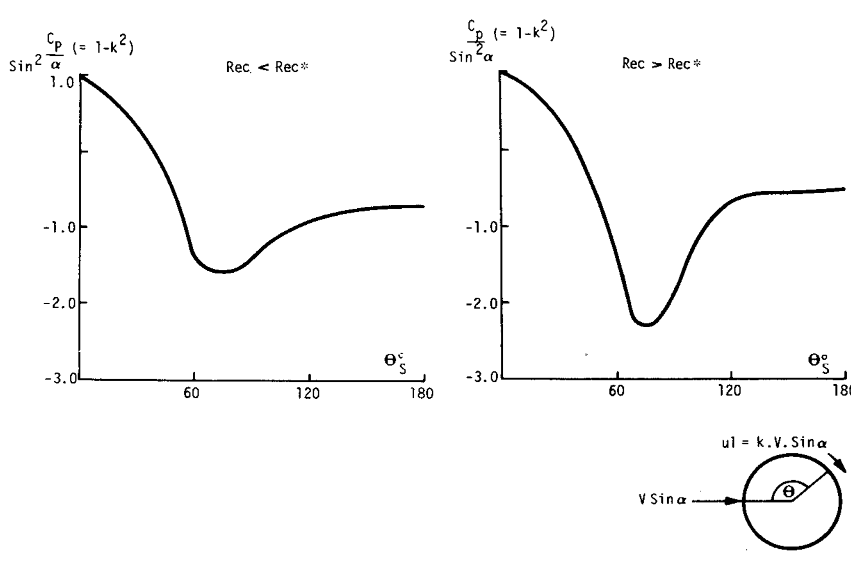

Fig. 3 Experimental k values for separated flow around cylindrical body sections.[8]

that, by analogy with the two-dimensional von Kàrmàn vortex street, vortex core axial velocity may be written

$$V_a = V \sin\alpha(\cot^2\alpha + \chi^2)^{1/2}$$

Measured values of χ were presented in Ref. 5 and are reproduced in Fig. 2. Further, for crossflow Mach numbers less than 0.75, g as measured in Ref. 5 is given by $5d/\tan\alpha$.

For convenience, U is related to the crossflow component ($V\sin\alpha$) of freestream velocity by $U = k.\ V.\sin\alpha$ (in two-dimensional potential theory, $k = 2\sin\theta$). Slender body theory[7] yields the result that the circumferential pressure coefficient C_p is given by

$$C_p/\sin^2\alpha = 1 - k^2$$

Adhering to this representation for k, experimental evidence from Ref. 8 is introduced from which $C_p/\sin^2\alpha$, and hence k, may be obtained as a function of θ (see Fig. 3). Note that k may be found whether the crossflow Reynolds number is subcritical or supercritical, i.e., less or greater than 10^5.

Finally, from Ref. 6, θ_s is obtained for a single Mach number (0.8) at various angles of attack for portions of the body where the wake is asymmetric, i.e., where θ_s is essentially constant with x (see Fig. 4). A possible difficulty here is that θ_s was obtained for $Re\,c > Re\,c^*$. However, it is not expected that the effect for cases where $Re\,c < Re\,c^*$ will be large.

With the preceding data available, and assuming that the two-dimensional result $\Lambda = \frac{1}{2}$ holds, Eq. (1) may be rewritten in terms of the strength parameter Γ_p as

$$\Gamma_p = \frac{\Gamma}{dV\sin\alpha} = \int_0^{g/d} k^2\, d\left(\frac{x}{d}\right) \Big/ 4(\cot^2\alpha + x^2)^{1/2} \qquad (2)$$

For regions of the body where the wake is asymmetric

$$\Gamma_p = k^2\left(\frac{g}{d}\right) \Big/ 4(\cot^2\alpha + \chi^2)^{1/2} \qquad (3)$$

i.e., θ_s, and hence k, are usually constant with x in such regions.

Γ_p is calculated as follows for $\alpha = 30°$, $M = 0.8$ ($M_c = 0.4$): from Fig. 4, $\theta_s = 95°$; from Fig. 3, $k^2 = 2.25$ ($Re\,c < Re\,c^*$, to match conditions of Ref. 5); from Ref. 5, $g/d = 5/\tan 30° = 8.66$; from Fig. 2, $\chi = 0.8$. Inputting these quantities to Eq. (3) yields

$$\Gamma_p = 2.56$$

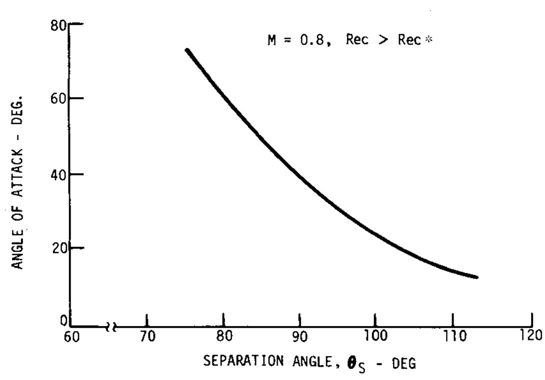

Fig. 4 Variation of separation angle with angle of attack.

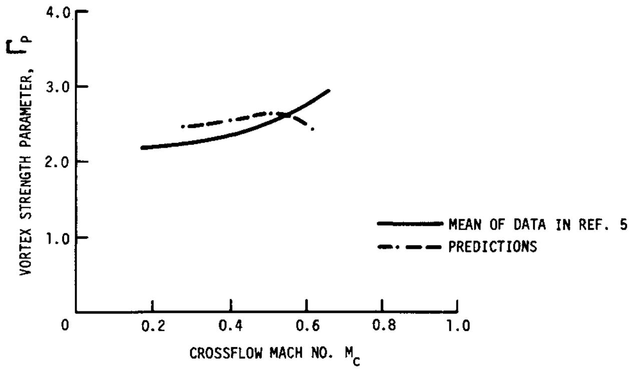

Fig. 5 Comparison of predicted and experimental Γ_p.

The corresponding mean value from Ref. 5 is 2.35. Further comparisons with data from Ref. 5 are shown in Fig. 5. Over-all matching is quite good.

Conclusions

The method presented for estimating the strength of wake vortices shed from slender bodies shows quite reasonable accuracy. This encouraging result indicates the possibility of handling further separated wake flows in the manner outlined, for example, vortices shed from low aspect ratio wing leading edges. The procedure appears, in principle, to offer a reasonable, short-term alternative to complete solution of the governing equations for such separated flows.

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